Appendix: Aircraft Dynamic Equations

The flight dynamics of rigid aircraft are described by the following set of eight first-order differential equations taken from Ref. 10:

$$\dot{M} = (1/mv_s)\left[T_m\eta\cos\alpha\cos\beta - \tfrac{1}{2}C_D(\alpha)\rho v_s^2 M^2 S - mg\sin\gamma\right]$$

$$\dot{\beta} = (1/mv_s M)\left[-T_m\eta\cos\alpha\sin\beta + \tfrac{1}{2}C_Y(\beta,\delta a,\delta r)\rho v_s^2 M^2 S + mg\sin\mu\cos\gamma\right] + (p\sin\alpha - r\cos\alpha)$$

$$\dot{\alpha} = q - (1/\cos\beta)\Big\{(p\cos\alpha + r\sin\alpha)\sin\beta + (1/mv_s M)\left[T_m\eta\sin\alpha + \tfrac{1}{2}C_L(\alpha,\delta e)\rho v_s^2 M^2 S - mg\cos\mu\cos\gamma\right]\Big\} \tag{A1}$$

$$\dot{p} = [(I_y - I_z)/I_x]qr + (1/2I_x)\rho(v_s M)^2 SbC_l(\alpha,\beta,p,r,\delta a,\delta r)$$

$$\dot{q} = [(I_z - I_x)/I_y]pr + (1/2I_y)\rho(v_s M)^2 ScC_m(\alpha,q,\delta e)$$

$$\dot{r} = [(I_x - I_y)/I_z]pq + (1/2I_z)\rho(v_s M)^2 SbC_n(\alpha,\beta,p,r,\delta a,\delta r) \tag{A2}$$

$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta$$

$$\dot{\theta} = q\cos\phi - r\sin\phi \tag{A3}$$

The wind axis orientation angles μ and γ are defined as follows:

$$\sin\gamma = \cos\alpha\cos\beta\sin\theta - \sin\beta\sin\phi\cos\theta - \sin\alpha\cos\beta\cos\phi\cos\theta$$

$$\sin\mu\cos\gamma = \sin\theta\cos\alpha\sin\beta + \sin\phi\cos\theta\cos\beta - \sin\alpha\sin\beta\cos\phi\cos\theta$$

$$\cos\mu\cos\gamma = \sin\theta\sin\alpha + \cos\alpha\cos\phi\cos\theta$$

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Near-Eigenaxis Rotation Control Law Design for Moving-to-Rest Maneuver

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Introduction

IT has been widely reported that the eigenaxis rotation maneuver reduces energy consumption of spacecraft.[1–3] Bilimoria and Wie[2] showed, however, that in general the eigenaxis rotation maneuver is not time optimal. Steyn[3] presented a near-minimum-time control technique for rotating a spacecraft around its eigenaxis. The applications of the mentioned eigenaxis rotation control laws have been confined to rest-to-rest maneuvers. The immediate eigenaxis rotation is impossible when the spacecraft is initially in motion, except for the case when the angular velocity is parallel to the eigenaxis.

In this Note, a new eigenaxis rotation algorithm, which is applicable to a spacecraft initially in motion, is suggested. Even when the initial angular velocity vector is not parallel to the eigenaxis, the algorithm causes the angular velocity vector to change its direction parallel to the instantaneous eigenaxis and then induces the eigenaxis rotation. Eigenaxis rotation occurs at all phases of the maneuver except for the start phase. Thus, the suggested algorithm is referred to as a near-eigenaxis rotation control law. However, a significant improvement in energy consumption can be achieved.

Design of Attitude Control Law

Consider first a control law design procedure for a large angle maneuver. Let the spacecraft states be represented by the angular velocity vector $\boldsymbol{\omega}$ and the four-dimensional Euler parameter vector $\bar{\boldsymbol{\beta}} = [\beta_0\ \beta_1\ \beta_2\ \beta_3]^T$. The system kinematic equation can then be given by[4]

$$\dot{\bar{\boldsymbol{\beta}}} = \frac{1}{2}\begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix}\boldsymbol{\omega} \equiv \frac{1}{2}G(\bar{\boldsymbol{\beta}})\boldsymbol{\omega} \tag{1}$$

The spacecraft is equipped with reaction wheels that serve as torque actuators. Dynamic equations can then be written as

$$I\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega}^\times I\boldsymbol{\omega} - \boldsymbol{\omega}^\times h - u \tag{2}$$

$$\dot{h} = u \tag{3}$$

where I is the inertia matrix and h and u are the wheel angular momentum and the control torque, respectively. Among the existing methods for nonlinear control law design, one of the most powerful is Lyapunov's approach (see Refs. 5 and 6). Let the target states for a spacecraft be $\boldsymbol{\omega}_f$ and $\bar{\boldsymbol{\beta}}_f$. Then, consider a Lyapunov function given by

$$V = (\bar{\boldsymbol{\beta}} - \bar{\boldsymbol{\beta}}_f)^T(\bar{\boldsymbol{\beta}} - \bar{\boldsymbol{\beta}}_f) + \tfrac{1}{2}k_2^{-1}(\boldsymbol{\omega} - \boldsymbol{\omega}_f)^T(\boldsymbol{\omega} - \boldsymbol{\omega}_f) \tag{4}$$

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where k_2^{-1} is a positive constant. It can be easily shown that the following control law enables the system to track the target states in a stable manner:

$$u = -k_2 I G^T(\bar{\beta})\bar{\beta}_f - \omega^\times I\omega - \omega^\times h - I\dot{\omega}_f + IK_1(\omega - \omega_f) \tag{5}$$

where K_1 is a positive definite gain matrix. When the target state is a point fixed at $\omega_f = 0$ and $\bar{\beta}_f = [1\ \ 0\ \ 0\ \ 0]^T$, the preceding control law becomes a simple regulator control law given by

$$u = -\omega^\times I\omega - \omega^\times h - I\dot{\omega}_f + IK_1\omega + k_2 I\beta \tag{6}$$

where β is an Euler parameter vector with reduced dimension as $\beta \equiv [\beta_1\ \ \beta_2\ \ \beta_3]^T$, that is, the eigenaxis vector. Then the closed-loop equation can be simplified as

$$\dot{\omega} + K_1\omega + k_2\beta = 0 \tag{7}$$

The motion governed by the preceding equation is the eigenaxis motion if $\omega(0) = 0$ or $\omega \| \beta(0)$ (Refs. 1–3). However, when the spacecraft is initially in motion and the $\omega(0)$ vector is not parallel to the initial eigenaxis vector β, then the resulting motion is not the eigenaxis rotation. To reduce the energy consumption, the eigenaxis maneuver portion needs to be maximized. As long as the angular velocity is parallel to the eigenaxis vector, the eigenaxis rotation can occur. Thus, the ω vector should be aligned with the β vector as quickly as possible. That is the basis idea of the new algorithm for the moving-to-rest maneuvers.

Moving-to-Rest Near-Eigenaxis Rotation Control Law Design

The angular velocity vector has two orthogonal components in the plane defined by ω and β, one component parallel to β and the other orthogonal to β as follows:

$$\omega = (\hat{\beta}\hat{\beta}^T)\omega + (\mathbf{1} - \hat{\beta}\hat{\beta}^T)\omega \tag{8}$$

where $\mathbf{1}$ is the 3×3 identity matrix and $\hat{\beta}$ is the normalized vector of β. The first term in the preceding equation represents the component parallel to β and the second term is orthogonal to β, as illustrated in Fig. 1. To start the eigenaxis motion, the angle θ between the vector ω and β should first be set zero. Rewrite Eq. (7) as

$$\dot{\omega} = -K_1\omega - k_2\beta \tag{9}$$

Referring to this equation, we can see that ω can be changed either by $K_1\omega$ or by $k_2\beta$. However, the directional change of ω is governed only by $K_1\omega$ term. The second term $k_2\beta$ contributes toward the convergence of the ω vector to the origin, and cannot influence the directional change of ω toward β. The vector $K_1\omega$ can affect both the convergence and directional change as shown in Eq. (8). Thus, if the gain matrix K_1 can be adjusted properly such that the orthogonal component of $K_1\omega$ is greater than the parallel

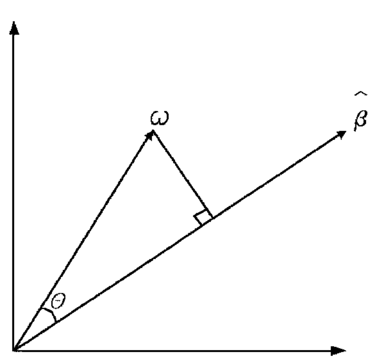

Fig. 1 Angular velocity components.

component of $K_1\omega$, then the rate of directional change of ω will be greater than the rate of convergence of ω to the origin. Then ω will be aligned parallel to β before it converges to the origin, and once the alignment is completed, the eigenaxis rotation commences. Therefore, for a moving-to-rest near-eigenaxis maneuver, we merely need to find a $K_1(t)$ that will quickly change the direction of ω toward β.

Using Eq. (8), the gain matrix $K_1(t)$ is selected as

$$K_1(t) = k_3\hat{\beta}\hat{\beta}^T + k_4(\mathbf{1} - \hat{\beta}\hat{\beta}^T) \tag{10}$$

where k_3 and k_4 are positive constants. Here, we can see that the rate of convergence of ω to the origin depends on the value of k_3 and that the rate of directional change of ω toward β depends on k_4. Thus, the ratio of the rate of directional change and the rate of the convergence depends on the relative magnitude of k_4 to k_3. For maximizing the eigenaxis rotation portion during the total maneuvering time span, the rate of directional change of ω toward β needs to be large. Therefore, the gain k_4 should be chosen greater than k_3.

Stability Analysis

The gain matrix K_1 in Eq. (10) should be positive definite for the asymptotic stability of the system. Even when K_1 is time varying, the system stability is guaranteed as long as K_1 is positive and bounded.[5,6] Therefore, we need to examine the boundedness and positive definiteness of the time varying $K_1(t)$.

Because $\hat{\beta}$ is a unit vector, $K_1(t)$ is bounded, and thus, the boundedness condition is met. The matrix K_1 is positive definite if the following condition is satisfied:

$$x^T K_1(t) x > 0, \qquad \text{for any} \qquad x \neq 0 \tag{11}$$

Referring to Eqs. (10) and (11), the gains k_3 and k_4 should satisfy the following condition:

$$(k_3 - k_4)x^T\hat{\beta}\hat{\beta}^T x + k_4 x^T x \geq 0 \tag{12}$$

We can see that $K_1(t)$ is positive definite when $k_3 > k_4$. However, the gain k_4 needs to be greater than k_3 for our purpose. Rewrite Eq. (12) as

$$k_4 x^T x - (k_4 - k_3)x^T\hat{\beta}\hat{\beta}^T x \geq 0 \tag{13}$$

If $k_4 > k_3 > 0$, then $K_1(t)$ is positive definite because $\|\hat{\beta}\hat{\beta}^T\| \leq 1$. Therefore, $K_1(t)$ is always positive definite, irrespective of the relative magnitude of k_4 and k_3. The asymptotic stability of the system is assured.

Applications of Moving-to-Rest Near-Eigenaxis Rotation Control Law

When using the time-varying gain matrix $K_1(t)$ in Eq. (10), the moving-to-rest near-eigenaxis control law becomes

$$u = -\omega^\times I\omega - \omega^\times h + [k_3\hat{\beta}\hat{\beta}^T + k_4(\mathbf{1} - \hat{\beta}\hat{\beta}^T)]I\omega + k_2 I\beta \tag{14}$$

The control law is applied to a model spacecraft with the following parameters:

$$I = \text{diag}(180, 80, 130)\ \text{kg} \cdot \text{m}^2$$

$$\bar{\beta}(0) = [0.61457\quad 0.2652\quad 0.2652\quad -0.6930]^T$$

$$\omega(0) = [1.4\quad 0.4\quad 0.2]^T\ \text{rad/s}$$

To compare the energy consumption performance of the eigenaxis maneuvers with general maneuvers, three cases are simulated. We select the gains of $k_2 = 0.4$ and $k_3 = 0.9$ for all three cases so that

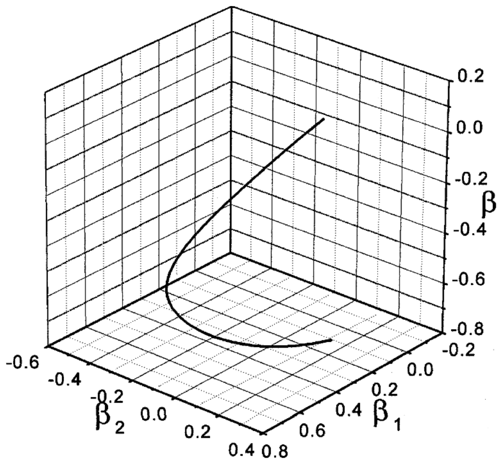

Fig. 2 General rotation.

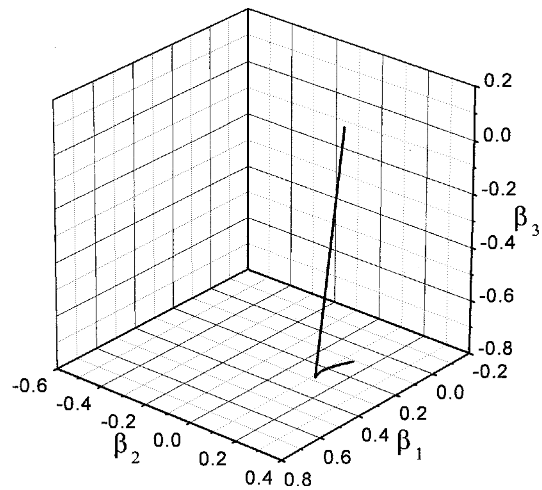

Fig. 4 Near-eigenaxis rotation: $k_3 = 0.9, k_4 = 5.4$ case.

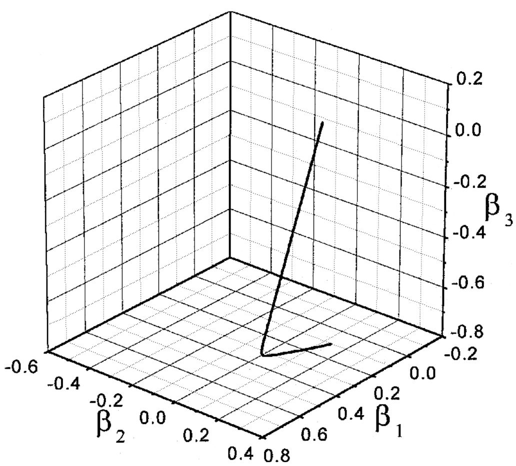

Fig. 3 Near-eigenaxis rotation: $k_3 = 0.9, k_4 = 2.7$ case.

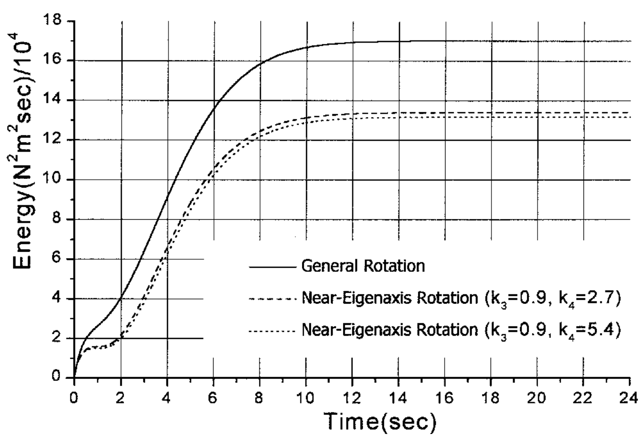

Fig. 5 Energy consumption.

the system will show the critically damped response during the convergence phase to the origin. The convergence time to the origin is thus identically about 18 s for all three cases.

Case 1: General Maneuver

If we select the same values for k_3 and k_4, then the gain matrix K_1 becomes time invariant, and a general maneuver occurs. We select the gains of $k_4 = k_3 = 0.9$. As shown in Fig. 2, the attitude vector β converges to the origin following a long maneuvering path as expected.

Cases 2 and 3: Near-Eigenaxis Maneuvers

For near-eigenaxis rotation maneuvers, two cases with $k_4 = 2.7$ and 5.4 are simulated to determine the effects of k_4 on the rate of directional change. For both cases, $k_2 = 0.4$ and $k_3 = 0.9$ as in the preceding case so that the convergence time to the origin is identical.

It is clearly shown that the value of k_4 affects the rate of directional change. For the gain k_4 set at $k_4 = 2.7$, the 2% settling time τ_s in directional change is estimated as 1.5 s. For $k_4 = 5.4$, however, the estimated settling time decreases to less than 0.7 s. Thus, we can see that the rate of directional change depends on k_4. Figures 3 and 4 show that compared with the general maneuver, paths to the origin are shorter in the sense of Euler parameter space for both cases. Whereas the total maneuvering takes about 18 s for both cases, the eigenaxis rotations are initiated after about 1.2 s for $k_4 = 5.4$ and after about 1.7 s for $k_4 = 2.7$. Figure 5 shows the comparison of the energy consumption for three cases. The energy consumption defined as

$$T = \frac{1}{2}\int u^T u \, \mathrm{d}t$$

decreases as k_4 increases.

Conclusions

A near-eigenaxis rotation algorithm is introduced, which is applicable to moving-to-rest maneuvers of spacecraft. The algorithm, just by using a time-varying gain matrix in the control law, causes the angular velocity vector to align itself parallel to the instantaneous eigenaxis, and then quickly induces the eigenaxis rotation. The eigenaxis rotation occurs during almost all phases of the maneuver except the start phase. A significant improvement in energy consumption can, thus, be achieved. The asymptotic stability of the new control law is also proved.

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Control Laws for Minimum Orbital Change—The Satellite Retrieval Problem

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Introduction

In a typical trajectory optimization problem the time of flight or propellant consumed is minimized, resulting in significant changes in one or more orbital elements. In situations where an upper-stage engine has failed to burn completely or a solid motor remains unfired, the optimization task is reversed; the goal is to expend as much propellant as possible, sometimes with little or no change in the orbital elements. In such cases the payload is often not in the desired orbit, and mission planners might want to retrieve the payload using the space shuttle and subsequently relaunch it. Safety concerns might prohibit returning the payload and upper stage in the shuttle cargo bay with propellants still onboard. This problem was first encountered in 1984 with the Westar and Palapa B spacecraft, which failed to achieve their intended orbits. The solid-propellant apogee kick motors were fired to deplete the propellants, and the spacecraft were safely retrieved.

Generally, there are no means to vent liquid propellants, and they must be consumed by reigniting the engine. The space shuttle itself must use this method in the event that one of the orbital maneuvering system pods is leaking propellant; standard procedure calls for consuming this leaking propellant by firing the thruster out of plane, thus minimizing the changes in semimajor axis and eccentricity. In this Note it is assumed that the vehicle is presently in an orbit compatible with shuttle operations for satellite retrieval. Using only a single sustained burn, the thrust generated needs to be directed so as to minimize the changes in semimajor axis a, eccentricity e, and inclination i, ensuring that the new orbit would be accessible by a later shuttle mission.

Solution Method

This method employs a combination of extremal control laws, each of which extremizes one orbital element. These control laws will also be straightforward to implement. Spencer and Culp[1] used this approach to generate control laws that maximized rates of change for several orbital elements individually on a sequence of discrete burn-arcs. Kluever and Oleson[2] blended the extremal control laws to change simultaneously several orbital elements in a low-thrust problem.

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This method requires control laws that each minimize a single orbital element's time derivative. For a satellite's retrieval, changes in its semimajor axis, eccentricity and inclination directly affect its accessibility by the shuttle. Changes in longitude of the ascending node, argument of periapsis, and time of periapsis passage are not as crucial because these can be accommodated through proper selection of shuttle launch time and subsequent orbital phasing.

Under the assumption of a spherical gravity field, satellite thrust results in the following rates for a, e, and i:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2a^2 v}{\mu} a_{\mathrm{TH}} \cos\theta \cos\sigma$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{a_{\mathrm{TH}}}{v}[2(e+\cos f)\cos\theta + (r/a)\sin f \sin\theta]\cos\sigma$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r}{\sqrt{\mu a(1-e^2)}}\cos(\omega + f) a_{\mathrm{TH}} \sin\sigma \tag{1}$$

where v is the velocity, μ is the gravitational parameter, a_{TH} is the thrust acceleration (assumed constant in this Note), θ is the thrust angle (measured between the velocity vector and the thrust vector's projection onto the orbital plane), f is the true anomaly, ω is the argument of periapsis, and σ is the angle between the orbital plane and the thrust vector.

Zero rate in a requires the steering angle

$$\theta_a^* = \pm\pi/2 \tag{2}$$

while zero rate in e requires

$$\theta_e^* = \tan^{-1}\left[\frac{2(e+\cos f)(1+e\cos f)}{(e^2-1)\sin f}\right] \tag{3}$$

and zero rate in i requires

$$\sigma_i^* = 0 \tag{4}$$

Using the RTN (radial, transverse, orbit-normal) coordinate system, the unit vectors giving the corresponding thrust direction for minimum rates of change in a, e, and i, respectively, are

$$\hat{u}_a = \left[\sin\left(\theta_a^* + \gamma\right), \cos\left(\theta_a^* + \gamma\right), 0\right]^T$$

$$\hat{u}_e = \left[\sin\left(\theta_e^* + \gamma\right), \cos\left(\theta_e^* + \gamma\right), 0\right]^T, \qquad \hat{u}_i = [u_{i,R}, u_{i,T}, 0]^T \tag{5}$$

where γ is the flight-path angle (measured from the local horizontal plane to the velocity vector). The radial and transverse components of $\hat{u}_i$ are irrelevant. Blending the control laws consists of forming a linear combination of $\hat{u}_a$, $\hat{u}_e$, and $\hat{u}_i$ with time-varying proportions, then normalizing to generate a new unit vector. Because minimizing i requires no out-of-plane thrusting, using that extremal control would effectively remove σ as a degree of freedom. Therefore, $\hat{u}_i$ is modified to permit out-of-plane thrusting: $\hat{u}_i = [0, 0, 1]^T$. The weighting function for i directly controls the amount of out-of-plane thrusting. This will allow the optimizer to find a solution that modulates the inclination. The unit vector is

$$\hat{u} = \frac{G_a(t)\hat{u}_a + G_e(t)\hat{u}_e + G_i(t)\hat{u}_i}{\|G_a(t)\hat{u}_a + G_e(t)\hat{u}_e + G_i(t)\hat{u}_i\|} \tag{6}$$

where the $G_x(t)$ are the weighting functions (gains) for the steering laws. This unit vector defines the thrust direction in the RTN system. The optimization consists of determining the $G_x(t)$ for a, e, and i in discretized form.